

Equal Superposition Transformations and Quantum Random Walks

Preeti Parashar

Physics and Applied Mathematics Unit

Indian Statistical Institute

203 B T Road, Kolkata 700 108, India

e-mail: parashar@isical.ac.in

Abstract

The largest ensemble of qubits which satisfy the general transformation of equal superposition is obtained by different methods, namely, linearity, no-superluminal signalling and non-increase of entanglement under LOCC. We also consider the associated quantum random walk and show that all unitary balanced coins give the same asymmetric spatial probability distribution. It is further illustrated that unbalanced coins, upon appropriate superposition, lead to new unbiased walks which have no classical analogues.

Keywords: Equal superposition ensemble; no-signalling; non-increase of entanglement; quantum random walk.

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1 Introduction

There has been considerable interest in the recent past to prove the non-existence of certain quantum unitary operations for arbitrary and unknown qubits. Some of the important ones are: the no-cloning theorem [1], the no-deleting principle [2], no-flipping operator [3] and the no-Hadamard operator [4]. These no-go theorems have been re-established by other physical fundamental principles, like the no-signalling condition and no-increase of entanglement under LOCC [5]-[10]. It is then natural to ask that if these operations do not work universally (i.e., for all qubits), then for what classes of quantum states it would be possible to perform a particular task by a single unitary operator. For example, the set of qubits which can be flipped exactly by the quantum NOT operator, lie on a great circle of the Bloch sphere [4, 11]. Likewise, the largest ensemble of states which can be rotated by the Hadamard gate was obtained in [12].

The Hadamard gate creates a superposition of qubit state and its orthogonal complement with equal amplitudes. In the present work, we consider the most general transformation where the superposition is with amplitudes which are equal upto a phase. In other words, the state and its orthogonal superimpose with equal probabilities but not necessarily with exactly the same amplitudes. First, we obtain the largest class of quantum states which can be superposed via this transformation. Second, it is shown, by using the no-signalling condition and non-increase of entanglement under LOCC, that this transformation does not hold for an arbitrary qubit.

The Hadamard transformation is known to be intimately connected to quantum random walks which were introduced in [13]. It has been used as a ‘coin flip’ transformation (balanced coin) to study the dynamics of such walks [14, 15]. In the same spirit, we consider the quantum random walk associated with our general transformation and study the probability distribution of the position of a particle. It is found that the entire family of such walks gives the same asymmetric distribution. We have also considered a unitary transformation with unequal amplitudes, serving as an unbalanced coin. It is shown that, after a suitable superposition, both types of coins lead to symmetric (unbiased) walks. However, in the case of unbalanced coin, we obtain new walks that have no classical analogues.

The paper is organized as follows: In Sec. 2 we present the equal superposition ensemble. Sec. 3 and 4 pertain to the proving of the non-existence of equal superposition transformation for an arbitrary qubit. If the state and its orthogonal could be superposed, then it must belong to the ensemble presented in Sec. I. This is achieved by imposing the condition of no-superluminal signalling and non-increase of entanglement under LOCC. Sec. 5 is devoted to the study of the associated quantum random walks. We end the paper with some conclusions in Sec. 6.

2 The equal superposition ensemble

The computational basis (CB) states $\{|0\rangle, |1\rangle\}$ of a qubit can be superposed most generally via the transformation

$$U|0\rangle \rightarrow \alpha|0\rangle + \beta|1\rangle, \quad U|1\rangle \rightarrow \gamma|0\rangle + \delta|1\rangle, \quad (1)$$

$\alpha, \beta, \gamma, \delta$ being arbitrary non-zero complex numbers. We are, however, interested in equal superposition (upto a phase) of the basis vectors. So let

$$\beta = e^{i\theta}\alpha, \quad \delta = e^{i\phi}\gamma. \quad (2)$$

The transformed states are required to be normalized and orthogonal to each other. This imposes the following constraints

$$\alpha\alpha^* = \gamma\gamma^* = 1/2, \quad \phi = \theta + \pi. \quad (3)$$

Eq.(1) then becomes

$$U|0\rangle \rightarrow \alpha|0\rangle + e^{i\theta}\alpha|1\rangle, \quad U|1\rangle \rightarrow \gamma|0\rangle - e^{i\theta}\gamma|1\rangle, \quad (4)$$

with the unitary matrix given by

$$U = \begin{bmatrix} \alpha & \gamma \\ e^{i\theta}\alpha & -e^{i\theta}\gamma \end{bmatrix}. \quad (5)$$

This gives an infinite family of transformations since θ can take any value between 0 and 2π , and α, γ are c - numbers satisfying the constraint (3). One can get rid of the overall factor by setting $\alpha = 1 = \gamma$ in Eq.(4). However, the states then become unnormalized. To restore normalization one could simply fix $\alpha = 1/\sqrt{2} = \gamma$. With this choice, the states in (4) reduce to the specific form of states lying on the equatorial great circle. So for the sake of generality, we shall refrain from assigning any particular value to these parameters.

Now, we address the following question: Which other orthogonal pair of qubit states $\{|\psi\rangle, |\bar{\psi}\rangle\}$ would transform under U in a similar manner as $\{|0\rangle, |1\rangle\}$? More precisely, we wish to find as to which class of qubits would satisfy

$$U|\psi\rangle \rightarrow \alpha|\psi\rangle + e^{i\theta}\alpha|\bar{\psi}\rangle, \quad U|\bar{\psi}\rangle \rightarrow \gamma|\psi\rangle - e^{i\theta}\gamma|\bar{\psi}\rangle, \quad \alpha\alpha^* = \gamma\gamma^* = 1/2. \quad (6)$$

For this purpose, we start with an arbitrary qubit state $|\psi\rangle$ and its orthogonal complement $|\bar{\psi}\rangle$ as a superposition of the CB states

$$|\psi\rangle = a|0\rangle + b|1\rangle, \quad |\bar{\psi}\rangle = b^*|0\rangle - a^*|1\rangle, \quad (7)$$

where the non-zero complex numbers obey the normalization condition $aa^* + bb^* = 1$. Substituting the above states in the first expression of Eq.(6) gives

$$U|\psi\rangle = (\alpha a + e^{i\theta}\alpha b^*)|0\rangle + (\alpha b - e^{i\theta}\alpha a^*)|1\rangle. \quad (8)$$

Assuming that U acts linearly on $|\psi\rangle$, we have

$$\begin{aligned} U|\psi\rangle &= aU|0\rangle + bU|1\rangle \\ &= (\alpha a + \gamma b)|0\rangle + (e^{i\theta}\alpha a - e^{i\theta}\gamma b)|1\rangle. \end{aligned} \quad (9)$$

Equating the coefficients in (8) and (9) gives

$$b = e^{i\theta}\frac{\alpha}{\gamma}b^*, \quad a + a^* = (e^{-i\theta} + \frac{\gamma}{\alpha})b = e^{-i\theta}b + e^{i\theta}b^* \quad (10)$$

Thus, we can state our main result:

The general equal superposition transformation (6) holds for all qubit pairs $\{|\psi\rangle, |\bar{\psi}\rangle\}$ which satisfy the constraint (10).

It can be explicitly checked that unitarity holds for these states. Consider two such distinct states $\{|\psi_1\rangle, |\psi_2\rangle\}$ and their orthogonal complements $\{|\bar{\psi}_1\rangle, |\bar{\psi}_2\rangle\}$ which transform according to (6). Taking the inner product, we have

$$\begin{aligned} \langle\psi_1|\psi_2\rangle &\rightarrow \alpha\alpha^*[\langle\psi_1|\psi_2\rangle + e^{i\theta}\langle\psi_1|\bar{\psi}_2\rangle + e^{-i\theta}\langle\bar{\psi}_1|\psi_2\rangle + \langle\bar{\psi}_1|\bar{\psi}_2\rangle], \\ \langle\bar{\psi}_1|\bar{\psi}_2\rangle &\rightarrow \gamma\gamma^*[\langle\psi_1|\psi_2\rangle - e^{i\theta}\langle\psi_1|\bar{\psi}_2\rangle - e^{-i\theta}\langle\bar{\psi}_1|\psi_2\rangle + \langle\bar{\psi}_1|\bar{\psi}_2\rangle], \end{aligned} \quad (11)$$

where $\alpha\alpha^* = \gamma\gamma^* = 1/2$. To see that these states actually satisfy the inner product relations, it is instructive to write the complex state parameters as $a = x + iy, b = u + iv$, where x, y, u, v are all real. In this notation, a state from this ensemble reads as

$$|\psi\rangle = \left\{\frac{1}{2}(e^{-i\theta} + \frac{\gamma}{\alpha})(u + iv) + iy\right\}|0\rangle + (u + iv)|1\rangle, \quad (12)$$

while its orthogonal would be

$$\begin{aligned} |\bar{\psi}\rangle &= (u - iv)|0\rangle - \left\{\frac{1}{2}(e^{i\theta} + \frac{\gamma^*}{\alpha^*})(u - iv) - iy\right\}|1\rangle \\ &= e^{-i\theta}\frac{\gamma}{\alpha}(u + iv)|0\rangle - \left\{\frac{1}{2}(e^{-i\theta} + \frac{\gamma}{\alpha})(u + iv) - iy\right\}|1\rangle. \end{aligned} \quad (13)$$

The inner product rules are then explicitly given as

$$\begin{aligned} \langle\psi_1|\psi_2\rangle &= \frac{1}{4}(6 + e^{i\theta}\frac{\gamma}{\alpha} + e^{-i\theta}\frac{\gamma^*}{\alpha^*})e^{-i\theta}\frac{\gamma}{\alpha}(u_1 + iv_1)(u_2 + iv_2) + y_1y_2 \\ &\quad + \frac{i}{2}(e^{-i\theta} + \frac{\gamma}{\alpha})[(u_1 + iv_1)y_2 - (u_2 + iv_2)y_1] = \langle\bar{\psi}_1|\bar{\psi}_2\rangle^*, \\ \langle\psi_1|\bar{\psi}_2\rangle &= ie^{-i\theta}\frac{\gamma}{\alpha}[(u_1 + iv_1)y_2 - (u_2 + iv_2)y_1] = -\langle\bar{\psi}_1|\psi_2\rangle^*. \end{aligned} \quad (14)$$

Substituting these in (11), we find that the inner product relations are indeed preserved.

Our result provides a very convenient unified framework to deduce any desired class of equally superposable quantum states. If the CB states obey a particular transformation (out of the infinite family (4)), then in a single shot we can obtain the entire ensemble

of qubits which would satisfy the same transformation. To demonstrate its usefulness, we present below, two known examples as special cases of our result.

1. *Hadamard ensemble* : Choose $\alpha = \frac{1}{\sqrt{2}}$, $\gamma = \frac{1}{\sqrt{2}}$, $\theta = 0$. Then ($U \rightarrow U_H$)

$$U_H|0\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle], \quad U_H|1\rangle = \frac{1}{\sqrt{2}}[|0\rangle - |1\rangle], \quad (15)$$

where

$$U_H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (16)$$

This is the well known Hadamard gate with its corresponding transformation. Notice that $U_H^2 = I$ since $U_H = [\sigma_x + \sigma_z]/\sqrt{2}$ where σ_x, σ_z are Pauli matrices. However, in general $U^2 \neq I$.

Further, substituting the above choice of the parameters, the constraint (10) gives $b = b^*$, i.e., b is real, and $a + a^* = 2b$, i.e., $Re(a) = b$. In terms of the real parameters x, y, u, v , the above deductions yield $v = 0$ and $u = x$. Therefore, the qubit states become restricted to

$$|\psi\rangle = (x + iy)|0\rangle + x|1\rangle, \quad |\bar{\psi}\rangle = x|0\rangle - (x - iy)|1\rangle, \quad 2x^2 + y^2 = 1. \quad (17)$$

Hence, we have obtained a special class of states which transform under the action of the Hadamard matrix U_H via the transformation

$$U_H|\psi\rangle = \frac{1}{\sqrt{2}}[|\psi\rangle + |\bar{\psi}\rangle], \quad U_H|\bar{\psi}\rangle = \frac{1}{\sqrt{2}}[|\psi\rangle - |\bar{\psi}\rangle]. \quad (18)$$

In other words, this proves the existence of the Hadamard gate (16) for any qubit chosen from the ensemble (17).

2. *Invariant ensemble* : Choose $\alpha = \frac{1}{\sqrt{2}}$, $\gamma = \frac{i}{\sqrt{2}}$, $\theta = \frac{\pi}{2}$. Then ($U \rightarrow U_I$)

$$U_I|0\rangle = \frac{1}{\sqrt{2}}[|0\rangle + i|1\rangle], \quad U_I|1\rangle = \frac{1}{\sqrt{2}}[i|0\rangle + |1\rangle], \quad (19)$$

where

$$U_I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}. \quad (20)$$

An interesting property of this transformation is that it goes into itself, i.e., $U_I|0\rangle \leftrightarrow U_I|1\rangle$ under the interchange $|0\rangle \leftrightarrow |1\rangle$. For this reason we shall refer to it as being ‘invariant’. The matrix U_I is symmetric but not hermitian and $U_I^2 = i\sigma_x$ (i.e., the NOT gate) since $U_I = [I + i\sigma_x]/\sqrt{2}$.

Now, in order to find as to which qubit states would satisfy

$$U_I|\psi\rangle = \frac{1}{\sqrt{2}}[|\psi\rangle + i|\bar{\psi}\rangle], \quad U_I|\bar{\psi}\rangle = \frac{1}{\sqrt{2}}[i|\psi\rangle + |\bar{\psi}\rangle] \quad (21)$$

we substitute the above values of α, γ , and θ in (10). This yields $b = b^*$, i.e., b is real, and $a + a^* = 0$, i.e., $Re(a) = 0$, implying that a is purely imaginary. Again assuming $a = x + iy$

and $b = u + iv$, these constraints give $v = 0$ and $x = 0$. Therefore, the qubit states become restricted to

$$|\psi\rangle = iy|0\rangle + u|1\rangle, \quad |\bar{\psi}\rangle = u|0\rangle + iy|1\rangle, \quad y^2 + u^2 = 1. \quad (22)$$

The above two ensembles were obtained in [12] by treating each one separately. Here we have shown that they can be deduced from a single general ensemble of equally superposed qubits.

The family of transformations which remain invariant under the interchange of $|0\rangle$ and $|1\rangle$ is a subset of the general family (4), and every member is essentially of the type (19). To see this let us consider the general transformation (4). For this to be invariant we must have $\alpha = -e^{i\theta}\gamma$ and $\gamma = e^{i\theta}\alpha$ which implies that $\theta = \pi/2, 3\pi/2$. Substituting $\gamma = \pm i\alpha$ in (4) we obtain the general form of the invariant transformation ($U \rightarrow U_I'$)

$$U_I'|0\rangle = \alpha[|0\rangle \pm i|1\rangle], \quad U_I'|1\rangle = \alpha[\pm i|0\rangle + |1\rangle], \quad \alpha\alpha^* = 1/2, \quad (23)$$

where

$$U_I' = \alpha \begin{bmatrix} 1 & \pm i \\ \pm i & 1 \end{bmatrix}. \quad (24)$$

Since α is an overall phase factor, it can be readily verified that every member of (23) would lead to exactly the same ensemble (22). Thus, (19) can be regarded as a representative of the invariant family (23). In what follows, we shall establish our main result in the context of two other physical principles, namely; the no-superluminal signalling condition and the non-increase of entanglement under LOCC.

3 No-superluminal signalling

Let us consider the CB states transforming via Eq.(4), and a qubit state $|\psi\rangle$ transforming under the same unitary matrix U via the first expression in (6). We first show that if $|\psi\rangle$ is completely arbitrary, then this would imply superluminal signalling. For this purpose, assume that Alice possesses a $3d$ qutrit while Bob has a $2d$ qubit and both share the following entangled state:

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{3}} (|0\rangle_A|0\rangle_B + |1\rangle_A|\psi\rangle_B + |2\rangle_A|1\rangle_B). \quad (25)$$

The density matrix of the combined system is defined as $\rho_{AB} = |\phi\rangle_{AB}\langle\phi|$. Alice's reduced density matrix can be obtained by tracing out Bob's part

$$\begin{aligned} \rho_A = \text{tr}_B(\rho_{AB}) &= \frac{1}{3} [|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| \\ &+ a|1\rangle\langle 0| + a^*|0\rangle\langle 1| + b|1\rangle\langle 2| + b^*|2\rangle\langle 1|]. \end{aligned} \quad (26)$$

Now Bob applies the above mentioned unitary transformation on his qubit states $\{|0\rangle, |1\rangle, |\psi\rangle\}$ in Eq.(25). But, he does not communicate any information to Alice regarding his operation.

The shared state then changes to

$$(I \otimes U)|\phi\rangle_{AB} = |\phi'\rangle_{AB} = \frac{1}{\sqrt{3}}[\alpha|00\rangle + e^{i\theta}\alpha|01\rangle + \alpha|1\psi\rangle + e^{i\theta}\alpha|1\bar{\psi}\rangle + \gamma|20\rangle - e^{i\theta}\gamma|21\rangle]. \quad (27)$$

After this operation, Alice's new reduced density matrix becomes

$$\begin{aligned} \rho'_A &= \frac{1}{3} [|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| \\ &+ \frac{1}{2}(a + e^{-i\theta}b + e^{i\theta}b^* - a^*)|1\rangle\langle 0| + \frac{1}{2}(a^* + e^{i\theta}b^* + e^{-i\theta}b - a)|0\rangle\langle 1| \\ &+ \alpha\gamma^*(a - e^{-i\theta}b + e^{i\theta}b^* + a^*)|1\rangle\langle 2| + \alpha^*\gamma(a^* - e^{i\theta}b^* + e^{-i\theta}b + a)|2\rangle\langle 1|]. \end{aligned} \quad (28)$$

Comparing the coefficients of each term in (26) and (28), it is evident that $\rho'_A \neq \rho_A$ for arbitrary choices of the parameters a and b . So, in principle, Alice can distinguish between ρ_A and ρ'_A , although Bob has not revealed anything to her about his operation. This implies that, with the help of entanglement, superluminal communication has taken place. But faster-than-light communication is forbidden by special theory of relativity. Hence, we conclude that the equal superposition transformation does not exist for an arbitrary qubit.

If, however, we impose that the no-signalling constraint should not be violated, then ρ_A and ρ'_A should be equal because the action of U is a trace preserving local operation performed only at Bob's side. Comparing coefficients of the term $|1\rangle\langle 0|$ in (26) and (28) we recover the condition $a + a^* = e^{-i\theta}b + e^{i\theta}b^*$. From $|1\rangle\langle 2|$ we have $\alpha\gamma^*(a - e^{-i\theta}b + e^{i\theta}b^* + a^*) = b$ which yields $2\alpha\gamma^*e^{i\theta}b^* = b$. Substituting $\gamma^* = \frac{1}{2\gamma}$, we get the other constraint $b = e^{i\theta}\frac{\alpha}{\gamma}b^*$. Thus, the no-signalling condition gives exactly the same class of states that was obtained initially from linearity.

4 Non-increase of entanglement under LOCC

Here we shall first show the non-existence of the unitary operation (6) for an arbitrary $|\psi\rangle$ by considering the fact that local operations and classical communication cannot increase the entanglement content of a quantum system. It turns out that, ρ_A and ρ'_A above, have equal eigenvalues $(0, 1/3, 2/3)$. This means that there is no change in entanglement before and after the unitary operation. So we consider a different shared resource which has been used in [9, 10] for studying flipping and Hadamard operations,

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{1+b^*b}}[|0\rangle_A \frac{|0\rangle_{B1}|1\rangle_{B2} - |1\rangle_{B1}|0\rangle_{B2}}{\sqrt{2}} + |1\rangle_A \frac{|0\rangle_{B1}|\psi\rangle_{B2} - |\psi\rangle_{B1}|0\rangle_{B2}}{\sqrt{2}}], \quad (29)$$

where the first qubit is with Alice while the other two are at Bob's side. Repeating the protocol, we obtain Alice's reduced density operator as

$$\rho_A = \frac{1}{1+b^*b}[|0\rangle\langle 0| + b^*b|1\rangle\langle 1| + b|1\rangle\langle 0| + b^*|0\rangle\langle 1|]. \quad (30)$$

The amount of entanglement given by the von Neumann entropy is zero since the eigenvalues of ρ_A are 0 and 1. This means that the resource state (29) is a product state in the A:B cut. Now Bob applies the trace preserving general transformation on the last particle (B2) in Eq.(29), which results in the state

$$|\Phi'\rangle_{AB} = \frac{1}{\sqrt{2N}} \left[\gamma|000\rangle - e^{i\theta}\gamma|001\rangle - \alpha|010\rangle - e^{i\theta}\alpha|011\rangle \right. \\ \left. + \alpha|10\psi\rangle + e^{i\theta}\alpha|10\bar{\psi}\rangle - \alpha|1\psi0\rangle - e^{i\theta}\alpha|1\psi1\rangle \right], \quad (31)$$

where $N = 2 + \frac{1}{4}\{(a - a^*)^2 - (e^{-i\theta}b + e^{i\theta}b^*)(a + a^*)\}$. Since a and b are arbitrary, so in general, the above state is entangled in the A:B cut. This implies that entanglement has been created by local operation. However, we know that entanglement cannot be increased by local operations even if classical communication is allowed. Therefore, the above contradiction leads us to the conclusion that the unitary operator (5) cannot perform the same task for an arbitrary qubit, as it does for the CB states $|0\rangle$ and $|1\rangle$.

We now derive the conditions under which the entanglement in the state would remain zero even after the application of U . For this purpose we have to compare the eigenvalues of the respective density matrices on Alice's side. So after Bob's operation

$$\rho'_A = \frac{1}{N} [|0\rangle\langle 0| + (N-1)|1\rangle\langle 1| + D|1\rangle\langle 0| + D^*|0\rangle\langle 1|], \quad (32)$$

where $D = \frac{1}{2}\{\alpha\gamma^*(a + a^* - e^{-i\theta}b + e^{i\theta}b^*) + b\}$. The eigenvalue equation of the above matrix gives two roots, namely,

$$\lambda_{\pm} = \frac{1}{2} \pm \frac{\sqrt{N^2 - 4(N-1-DD^*)}}{2N}, \quad (33)$$

In order to maintain the same amount of entanglement in the system before and after the unitary operation, we should equate these two roots λ_{\pm} of ρ'_A to the eigenvalues 0 and 1 of ρ_A . This furnishes the constraint $DD^* = N - 1$. Substituting the expressions for N , D and D^* , and rearranging the terms, this condition acquires the form

$$[(a + a^*) - (e^{-i\theta}b + e^{i\theta}b^*)][\frac{1}{4}(a + a^*) + \frac{1}{4}(e^{-i\theta}b + e^{i\theta}b^*) + \gamma\alpha^*b + \alpha\gamma^*b^*] \\ + 2(e^{-i\theta}\gamma\alpha^*b^2 + e^{i\theta}\alpha\gamma^*b^{*2} + bb^*) = 4 + (a - a^*)^2 - (e^{-i\theta}b + e^{i\theta}b^*)(a + a^*). \quad (34)$$

Using $\alpha^* = \frac{1}{2\alpha}$, $\gamma^* = \frac{1}{2\gamma}$ on the L.H.S. and adding and subtracting $2aa^*$ on the R.H.S., the above relation is recast as

$$[(a + a^*) - (e^{-i\theta}b + e^{i\theta}b^*)][\frac{1}{4}(a + a^*) + \frac{1}{4}(e^{-i\theta}b + e^{i\theta}b^*) + \frac{1}{2}(\frac{\gamma}{\alpha}b + \frac{\alpha}{\gamma}b^*)] \\ + [e^{-i\theta/2}\sqrt{\frac{\gamma}{\alpha}}b + e^{i\theta/2}\sqrt{\frac{\alpha}{\gamma}}b^*]^2 = 4bb^* + [(a + a^*) - (e^{-i\theta}b + e^{i\theta}b^*)](a + a^*) \quad (35)$$

which can be written more compactly as

$$\begin{aligned} & [(a + a^*) - (e^{-i\theta}b + e^{i\theta}b^*)][-\frac{3}{4}(a + a^*) + \frac{1}{4}(e^{-i\theta}b + e^{i\theta}b^*) + \frac{1}{2}(\frac{\gamma}{\alpha}b + \frac{\alpha}{\gamma}b^*)] \\ &= -[e^{-i\theta/2}\sqrt{\frac{\gamma}{\alpha}}b - e^{i\theta/2}\sqrt{\frac{\alpha}{\gamma}}b^*]^2. \end{aligned} \quad (36)$$

For convenience let us denote the two terms on L.H.S. by A and B . Then

$$[A][B] = -[e^{-i\theta/2}\sqrt{\frac{\gamma}{\alpha}}b - e^{i\theta/2}\sqrt{\frac{\alpha}{\gamma}}b^*]^2. \quad (37)$$

In the above, R.H.S. is either a real positive definite quantity or zero. For the L.H.S. to be positive, there, however, exist two possibilities:

(i) $A > 0, B > 0$: If we suppose that both terms in A are positive (they are already real), then $a + a^* = (e^{-i\theta}b + e^{i\theta}b^*) + C$, where C is a real positive constant. Thus $B > 0$ if $(\frac{\gamma}{\alpha}b + \frac{\alpha}{\gamma}b^*) > (e^{-i\theta}b + e^{i\theta}b^*) + \frac{3}{2}C$, which certainly is possible. Similarly if we suppose that both terms in A are negative, then $A > 0$ implies $|e^{-i\theta}b + e^{i\theta}b^*| = |a + a^*| + C$. Thus $B > 0$ if $(\frac{\gamma}{\alpha}b + \frac{\alpha}{\gamma}b^*) + |a + a^*| > \frac{C}{2}$.

(ii) $A < 0, B < 0$: In a similar manner, restrictions can be obtained for this case.

When R.H.S. is identically zero, then Eq.(37) would be satisfied uniquely if $A = 0, B = 0$. This gives $a + a^* = e^{-i\theta}b + e^{i\theta}b^*$ and $b = e^{i\theta}\frac{\alpha}{\gamma}b^*$, which are exactly the constraints that we have earlier obtained by linearity and no-signalling. It can be easily checked that the other cases $A = 0, B \neq 0$ and $A \neq 0, B = 0$ cannot exist due to the constraint fixed by R.H.S. being zero.

The above analysis demonstrates the possibility of existence of more solutions from the principle of non-increase of entanglement under LOCC. In the case of Hadamard operation, we had obtained a unique solution [10] from linearity, no-signalling and non-increase of entanglement under LOCC. Here we get a larger set of states with zero entanglement, from the last method. However, we must remember that we are looking for orthogonal pairs of states $\{|\psi\rangle, |\bar{\psi}\rangle\}$ which transform under the unitary operation defined by (6). In the above, we have considered only $|\psi\rangle$. Therefore, we must now carry out a similar analysis with $|\bar{\psi}\rangle$. More precisely, we take the set of qubit states $\{|0\rangle, |1\rangle, |\bar{\psi}\rangle\}$ and the shared state as

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{1+a^*a}}[|0\rangle_A \frac{|0\rangle_{B1}|1\rangle_{B2} - |1\rangle_{B1}|0\rangle_{B2}}{\sqrt{2}} + |1\rangle_A \frac{|0\rangle_{B1}|\bar{\psi}\rangle_{B2} - |\bar{\psi}\rangle_{B1}|0\rangle_{B2}}{\sqrt{2}}]. \quad (38)$$

Then Alice's reduced matrix reads as

$$\rho_A = \frac{1}{1+a^*a}[|0\rangle\langle 0| + a^*a|1\rangle\langle 1| - a^*|1\rangle\langle 0| - a|0\rangle\langle 1|]. \quad (39)$$

Bob now applies U on the states $\{|0\rangle, |1\rangle, |\bar{\psi}\rangle\}$ of his last qubit, thereby changing the shared state to

$$\begin{aligned} |\Psi'\rangle_{AB} = \frac{1}{\sqrt{2\mathcal{N}}} [& \gamma|000\rangle - e^{i\theta}\gamma|001\rangle - \alpha|010\rangle - e^{i\theta}\alpha|011\rangle \\ & + \gamma|10\psi\rangle - e^{i\theta}\gamma|10\bar{\psi}\rangle - \alpha|1\bar{\psi}0\rangle - e^{i\theta}\alpha|1\bar{\psi}1\rangle], \end{aligned} \quad (40)$$

where $\mathcal{N} = 2 - \frac{1}{2}\{\gamma\alpha^*b(a + a^* + e^{-i\theta}b - e^{i\theta}b^*) + \alpha\gamma^*b^*(a + a^* - e^{-i\theta}b + e^{i\theta}b^*)\}$. The corresponding reduced density matrix at Alice's end becomes

$$\rho'_A = \frac{1}{\mathcal{N}}[|0\rangle\langle 0| + (\mathcal{N} - 1)|1\rangle\langle 1| + \mathcal{D}|1\rangle\langle 0| + \mathcal{D}^*|0\rangle\langle 1|], \quad (41)$$

where $\mathcal{D} = \{\frac{1}{4}(a - a^* - e^{-i\theta}b - e^{i\theta}b^*) - \frac{a^*}{2}\}$. Like the previous case, this matrix has the following two eigenvalues

$$\lambda_{\pm} = \frac{1}{2} \pm \frac{\sqrt{\mathcal{N}^2 - 4(\mathcal{N} - 1 - \mathcal{D}\mathcal{D}^*)}}{2\mathcal{N}}. \quad (42)$$

Equating λ_{\pm} to the eigenvalues 0 and 1 of ρ_A gives the constraint $\mathcal{D}\mathcal{D}^* = \mathcal{N} - 1$ which can be expanded as

$$\begin{aligned} & [(a + a^*) - (e^{-i\theta}b + e^{i\theta}b^*)][-\frac{3}{8}(a + a^*) - \frac{1}{8}(e^{-i\theta}b + e^{i\theta}b^*) + \frac{1}{2}(\frac{\gamma}{\alpha}b + \frac{\alpha}{\gamma}b^*)] \\ & = -[e^{-i\theta/2}\sqrt{\frac{\gamma}{\alpha}}b - e^{i\theta/2}\sqrt{\frac{\alpha}{\gamma}}b^*]^2. \end{aligned} \quad (43)$$

Interestingly, this is a new restriction on the expression on R.H.S. This has to be consistent with the earlier restriction (36). Therefore equating (43) with (36) renders $a + a^* = e^{-i\theta}b + e^{i\theta}b^*$. Substituting this in either (36) or (43) yields $b = e^{i\theta}\frac{\alpha}{\gamma}b^*$. Thus we finally obtain a unique solution which is exactly the constraint (10) that defines our equal superposition ensemble.

We remark that such a situation was not encountered in the case of the Hadamard operation [10]. The reason is that the Hadamard transformation on $|\bar{\psi}\rangle$ is not independent since it can be obtained from the Hadamard transformation on the states $\{|0\rangle, |1\rangle, |\psi\rangle\}$ by using the special property of the Hadamard operator, namely, $U_H^2 = I$. However, in the present scenario (and in general), $U|\bar{\psi}\rangle$ cannot be deduced from $\{U|0\rangle, U|1\rangle, U|\bar{\psi}\rangle\}$. So it is necessary to take $U|\bar{\psi}\rangle$ into consideration, although whether this would provide some new restriction or not depends on the particular situation. For example, if we proceed with $|\bar{\psi}\rangle$, then linearity and no-signalling give nothing new but the same constraint (10) which was obtained from $|\psi\rangle$. However, in the framework of non-increase of entanglement under LOCC, this indeed yields a different condition (43), thereby forcing the set of solutions to a single unique solution. In view of the above, we are now in a position to make a stronger statement regarding our main result:

Any pair of qubit states $\{|\psi\rangle, |\bar{\psi}\rangle\}$ can be equally superposed via the unitary operation (6) if and only if they satisfy the constraint (10).

5 Quantum Random Walks

In the previous sections, we have obtained by different methods the class of qubit states which transform under the action of the unitary matrix U in a manner similar to Eq.

(4). As an application of this transformation (4), we are now going to study the quantum random walk associated with it. A particularly nice detailed survey of quantum walks has been given by Kempe [16], while [17] is a short review devoted to their applications to algorithms. The Hadamard matrix U_H has been widely used as a balanced coin (translation to the left or to the right with equal probability) to study the properties of a discrete-time quantum random walk (QRW)[15]. For example, the probability of finding the particle at a particular site after T steps of the walk have been investigated in detail. The Hadamard coin gives an asymmetric probability distribution for the QRW on a $1d$ line. This is because the Hadamard coin treats the two CB states differently; it multiplies the phase by -1 only in the case of $|1\rangle$. It has also been pointed out [16, 17] that if the Hadamard coin is replaced with the more symmetric coin U_I , then the probability distribution becomes symmetric. However, our analysis shows that this is not the case, even though U_I treats both $|0\rangle$ and $|1\rangle$ in a symmetrical way. This also motivates us to investigate the discrete-time QRW from a more general point of view. We shall study the behaviour of the walk by taking the general unitary matrix U given by Eq.(5) as our balanced coin. Subsequently, we shall comment on some interesting features that these walks share.

Consider a particle localized at position z on a $1d$ line. The Hilbert space \mathcal{H}_P is spanned by basis states $|z\rangle$, where z is an integer. This position Hilbert space is augmented by a coin space \mathcal{H}_C spanned by the two CB states $|0\rangle$ and $|1\rangle$. To avoid confusion with the position states, we now introduce a change of notation, and instead denote the CB states as $|\uparrow\rangle$ and $|\downarrow\rangle$. The total state of the particle lies in the Hilbert space $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_P$.

The first step of the random walk is a rotation in the coin space. We follow a procedure similar to what was adopted for the Hadamard walk [16]. In our general scenario, the matrix U given by (5) serves as the coin, with the following action (cf. Eq.(4))

$$U|\uparrow\rangle = \alpha|\uparrow\rangle + e^{i\theta}\alpha|\downarrow\rangle, \quad U|\downarrow\rangle = \gamma|\uparrow\rangle - e^{i\theta}\gamma|\downarrow\rangle. \quad (44)$$

The rotation is followed by translation with the application of the unitary operator

$$S = |\uparrow\rangle\langle\uparrow| \otimes \sum_z |z+1\rangle\langle z| + |\downarrow\rangle\langle\downarrow| \otimes \sum_z |z-1\rangle\langle z| \quad (45)$$

in the position space \mathcal{H}_P . Note that S is a ‘conditional’ translation operator since it moves the particle by one unit to the right if the coin state is $|\uparrow\rangle$, and to the left if it is $|\downarrow\rangle$

$$S|\uparrow\rangle \otimes |z\rangle = |\uparrow\rangle \otimes |z+1\rangle, \quad S|\downarrow\rangle \otimes |z\rangle = |\downarrow\rangle \otimes |z-1\rangle. \quad (46)$$

The particle is subjected to these two alternating unitary transformations. Therefore, the QRW of T steps is defined as the transformation \mathcal{A}^T , where \mathcal{A} acts on the total Hilbert space \mathcal{H} and is given by

$$\mathcal{A} = S(U \otimes I) \quad (47)$$

To start with, let the particle be in the $|\uparrow\rangle$ coin state and located at the position 0. Thus the total initial state is denoted by $|\phi\rangle = |\uparrow\rangle \otimes |0\rangle$. Let us now evolve the walk, for a few

steps, under successive action of the operator \mathcal{A} :

$$\begin{aligned}
|\phi\rangle &\rightarrow \alpha|\uparrow\rangle \otimes |1\rangle + e^{i\theta}\alpha|\downarrow\rangle \otimes |-1\rangle \\
&\rightarrow \alpha^2|\uparrow\rangle \otimes |2\rangle + e^{i\theta}(\alpha^2|\downarrow\rangle + \alpha\gamma|\uparrow\rangle) \otimes |0\rangle - e^{2i\theta}\alpha\gamma|\downarrow\rangle \otimes |-2\rangle \\
&\rightarrow \alpha^3|\uparrow\rangle \otimes |3\rangle + e^{i\theta}(\alpha^3|\downarrow\rangle + 2\alpha^2\gamma|\uparrow\rangle) \otimes |1\rangle - e^{2i\theta}\alpha\gamma^2|\uparrow\rangle \otimes |-1\rangle \\
&\quad + e^{3i\theta}\alpha\gamma^2|\downarrow\rangle \otimes |-3\rangle \\
&\rightarrow \alpha^4|\uparrow\rangle \otimes |4\rangle + e^{i\theta}(\alpha^4|\downarrow\rangle + 3\alpha^3\gamma|\uparrow\rangle) \otimes |2\rangle + e^{2i\theta}(\alpha^3\gamma|\downarrow\rangle - \alpha^2\gamma^2|\uparrow\rangle) \otimes |0\rangle \\
&\quad - e^{3i\theta}(\alpha^2\gamma^2|\downarrow\rangle - \alpha\gamma^3|\uparrow\rangle) \otimes |-2\rangle - e^{4i\theta}\alpha\gamma^3|\downarrow\rangle \otimes |-4\rangle
\end{aligned} \tag{48}$$

After T iterations, the particle is in an entangled state, say $|\phi_T\rangle$. The probability of finding the particle at a particular site z is given by

$$P_z = |(\langle\uparrow| \otimes \langle z|)|\phi_T\rangle|^2 + |(\langle\downarrow| \otimes \langle z|)|\phi_T\rangle|^2. \tag{49}$$

Let us analyze, step by step, the spatial probability distribution of the walk.

After $T = 1$: If we measure the position space after the first step, then the particle can be found at the site 1 with probability $\alpha\alpha^*$ and at the site -1 with the same probability. Since we already know that $\alpha\alpha^* = 1/2$ (normalization), so the particle moves with equal probability, one step to the right and one to the left of its original position. The walk is therefore, unbiased, just like the usual Hadamard walk.

After $T = 2$: The probabilities of finding the particle at positions 2, -2 and 0 are respectively,

$$P_2 = \alpha^2\alpha^{2*} = (\alpha\alpha^*)^2 = 1/4, \quad P_{-2} = \alpha\alpha^*\gamma\gamma^* = 1/4, \quad P_0 = \alpha^2\alpha^{2*} + \alpha\alpha^*\gamma\gamma^* = 1/2. \tag{50}$$

This step is also similar to the case of classical walk since the P 's are symmetrically distributed.

After $T = 3$: The distribution is

$$\begin{aligned}
P_3 &= \alpha^3\alpha^{3*} = (\alpha\alpha^*)^3 = 1/8, \quad P_{-3} = \alpha\alpha^*\gamma^2\gamma^{2*} = 1/8, \\
P_1 &= (\alpha\alpha^*)^3 + 4(\alpha\alpha^*)^2\gamma\gamma^* = 5/8, \quad P_{-1} = \alpha\alpha^*(\gamma\gamma^*)^2 = 1/8.
\end{aligned} \tag{51}$$

After the third step, the quantum walk begins to deviate from its classical counterpart. Although $P_3 = P_{-3}$, note that $P_1 \neq P_{-1}$. So the walk starts to be asymmetric, drifting towards the right since the site 1 has greater probability.

After $T = 4$: Similarly, upon measuring the position space after four iterations, we get the following asymmetric distribution

$$P_4 = 1/16, \quad P_{-4} = 1/16, \quad P_2 = 5/8, \quad P_{-2} = 1/8, \quad P_0 = 1/8. \tag{52}$$

Again, this differs from the symmetric classical probability distribution $P_4 = 1/16$, $P_{-4} = 1/16$, $P_2 = 1/4$, $P_{-2} = 1/4$, $P_0 = 3/8$. Proceeding in a similar way one can check the veracity of the foregoing conclusions by considering more steps of iterations. Clearly, the parameter θ which appears in the phase factor does not contribute to the probabilities. Also

since $\alpha\alpha^* = \gamma\gamma^*$, so for the purpose of probability distribution, only one of the parameters may be regarded as independent.

It is observed that the spatial probability distribution of the QRW corresponding to the general matrix U is asymmetrical and coincides exactly with that of the already known Hadamard walk. This means that every unitary transformation in which the qubit CB states are equally weighted, leads to the same probability distribution if the particle is taken in the same initial state. Therefore, we infer that even the symmetric coin U_I induces an asymmetrical walk. In fact, it can be argued easily as to why a symmetric probability distribution for the initial state $|\uparrow\rangle \otimes |0\rangle$ (or $|\downarrow\rangle \otimes |0\rangle$) is impossible. Let us refer to the distribution (51) after three iterations. If we want to make it symmetric, we must have $P_1 = P_{-1}$. This implies that

$$\alpha\alpha^*[(\alpha\alpha^*)^2 + 4\alpha\alpha^*\gamma\gamma^* - (\gamma\gamma^*)^2] = 0. \quad (53)$$

This equation cannot be satisfied since we know that $\alpha\alpha^* = 1/2$, and the term in the bracket equals 1. So L.H.S. can never be zero.

The direction of drift in the walk depends on the initial coin state and the bias is the result of quantum interference. So phases play a very crucial role in inducing asymmetry. This bias can, however, be taken care of if we again allow interference, so that the effect of the earlier superposition is negated. Thus, in order to make the walk symmetric or unbiased, we must take a superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$ as the initial coin state. However, we shall not assume apriori that $|\uparrow\rangle$ and $|\downarrow\rangle$ are superimposed with equal probability. For our general approach, we shall rather superimpose them with arbitrary amplitudes and obtain restrictions under which we can get a symmetric distribution. So we start the walk in the state

$$|\phi'\rangle = (x|\uparrow\rangle + y|\downarrow\rangle) \otimes |0\rangle, \quad xx^* + yy^* = 1 \quad (54)$$

(x and y are, in general, complex numbers) and let it evolve under the repeated action of the operator \mathcal{A} , as was done earlier.

After $T = 1$, the state becomes

$$\begin{aligned} |\phi_1\rangle &= (x\alpha + y\gamma)|\uparrow\rangle \otimes |1\rangle + e^{i\theta}(x\alpha - y\gamma)|\downarrow\rangle \otimes |-1\rangle \\ &= A|\uparrow\rangle \otimes |1\rangle + e^{i\theta}B|\downarrow\rangle \otimes |-1\rangle \end{aligned} \quad (55)$$

We now demand that the particle should be found at sites 1 and -1 with equal probability. This gives the constraint

$$AA^* = BB^* = 1/2 \quad (56)$$

which can be recast in terms of the transformation parameters as

$$xy^*\alpha\gamma^* + x^*y\alpha^*\gamma = 0. \quad (57)$$

Clearly, this holds only if $xy^*\alpha\gamma^*$ is purely imaginary.

After $T = 2$, the state of the particle becomes

$$|\phi_2\rangle = \alpha A|\uparrow\rangle \otimes |2\rangle + e^{i\theta}(\alpha A|\downarrow\rangle + \gamma B|\uparrow\rangle) \otimes |0\rangle - e^{2i\theta}\gamma B|\downarrow\rangle \otimes |-1\rangle \quad (58)$$

and the probabilities are

$$P_2 = \alpha\alpha^*AA^* = 1/4, \quad P_{-2} = \gamma\gamma^*BB^* = 1/4, \quad P_0 = \alpha\alpha^*AA^* + \gamma\gamma^*BB^* = 1/2. \quad (59)$$

After $T = 3$, the state evolves into

$$\begin{aligned} |\phi_3\rangle &= \alpha^2 A|\uparrow\rangle \otimes |3\rangle + e^{i\theta}(\alpha^2 A|\downarrow\rangle + 2x\alpha^2\gamma|\uparrow\rangle) \otimes |1\rangle \\ &- e^{2i\theta}(\gamma^2 B|\uparrow\rangle + 2y\gamma^2\alpha|\downarrow\rangle) \otimes |-1\rangle + e^{3i\theta}\gamma^2 B|\downarrow\rangle \otimes |-3\rangle \end{aligned} \quad (60)$$

The probabilities for the odd sites are

$$\begin{aligned} P_3 &= (\alpha\alpha^*)^2 AA^* = 1/8, \quad P_{-3} = (\gamma\gamma^*)^2 BB^* = 1/8 \\ P_1 &= (\alpha\alpha^*)^2 AA^* + 4xx^*(\alpha\alpha^*)^2\gamma\gamma^* = 1/8 + xx^*/2 \\ P_{-1} &= (\gamma\gamma^*)^2 BB^* + 4yy^*(\gamma\gamma^*)^2\alpha\alpha^* = 1/8 + yy^*/2 \end{aligned} \quad (61)$$

For symmetry, $P_1 = P_{-1}$ which, in turn, implies that $xx^* = yy^*$. But from normalization, we have $xx^* + yy^* = 1$. This restricts the value to $xx^* = yy^* = 1/2$. So the two amplitudes are equal, upto a phase factor, leading to an equal superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$. We have thus found that in order to make the quantum walk associated with the matrix U symmetric, it is necessary to take an ‘equally’ superposed coin state in such a way that $xy^*\alpha\gamma^*$ is purely imaginary. We present a new example to illustrate this situation.

Example: Choose $\alpha = \gamma = \frac{1+i}{2}$ and $\theta = \frac{3\pi}{2}$. The transformation (44) becomes

$$U|\uparrow\rangle = \frac{1+i}{2}(|\uparrow\rangle - i|\downarrow\rangle), \quad U|\downarrow\rangle = \frac{1+i}{2}(|\uparrow\rangle + i|\downarrow\rangle) \quad (62)$$

This has the features of a ‘hybrid’ between the Hadamard and the Invariant transformations discussed earlier. As expected, this gives an asymmetric walk. However, if we take the coin state in a superposition with amplitudes $x = \frac{1}{\sqrt{2}}$ and $y = \frac{-i}{\sqrt{2}}$, then the condition that $xy^*\alpha\gamma^*$ is purely imaginary is satisfied. Thus upon evolving the walk with the initial state $\frac{1}{\sqrt{2}}(|\uparrow\rangle - i|\downarrow\rangle) \otimes |0\rangle$, we do get the symmetric probability distribution which coincides with the classical one.

Unbalanced coin:

We have seen above that unitary balanced coins lead to a symmetric walk after appropriate superposition. Now we shall show that even unbalanced coins can yield an unbiased walk under similar restrictions.

Consider the unequal superposition transformation given in [12]

$$\mathcal{U}|\uparrow\rangle = p|\uparrow\rangle + q|\downarrow\rangle, \quad \mathcal{U}|\downarrow\rangle = q^*|\uparrow\rangle - p^*|\downarrow\rangle, \quad pp^* + qq^* = 1, \quad (63)$$

where, in general, $pp^* \neq qq^*$ and the unitary matrix \mathcal{U} is given by

$$\mathcal{U} = \begin{bmatrix} p & q^* \\ q & -p^* \end{bmatrix}. \quad (64)$$

Following the same procedure, let us start the walk in the superposed state

$$|\Phi'\rangle = (r|\uparrow\rangle + s|\downarrow\rangle) \otimes |0\rangle, \quad rr^* + ss^* = 1, \quad (65)$$

r, s being non-zero c - numbers. After $T = 1$, the state becomes

$$\begin{aligned} |\Phi_1\rangle &= (rp + sq^*)|\uparrow\rangle \otimes |1\rangle + (rq - sp^*)|\downarrow\rangle \otimes |-1\rangle \\ &= E|\uparrow\rangle \otimes |1\rangle + F|\downarrow\rangle \otimes |-1\rangle. \end{aligned} \quad (66)$$

For a symmetric probability distribution, we must have

$$EE^* = FF^* = 1/2, \quad (67)$$

which leads to the constraint

$$rs^*pq + r^*sp^*q^* = 0, \quad rr^* = ss^*. \quad (68)$$

This implies that like the previous case, here also the coin state must be in *equal* superposition (upto a phase) of $|\uparrow\rangle$ and $|\downarrow\rangle$.

After $T = 2$, the particle is in the entangled state

$$|\Phi_2\rangle = pE|\uparrow\rangle \otimes |2\rangle + (qE|\downarrow\rangle + q^*F|\uparrow\rangle) \otimes |0\rangle - p^*F|\downarrow\rangle \otimes |-2\rangle \quad (69)$$

with the probability distribution

$$P_2 = pp^*EE^* = \frac{1}{2}pp^*, \quad P_{-2} = pp^*FF^* = \frac{1}{2}pp^*, \quad P_0 = qq^*EE^* + qq^*FF^* = qq^*, \quad (70)$$

After $T = 3$, the entangled state is read as

$$\begin{aligned} |\Phi_3\rangle &= p^2E|\uparrow\rangle \otimes |3\rangle + pqE|\downarrow\rangle \otimes |1\rangle + (qq^*E + pq^*F)|\uparrow\rangle \otimes |1\rangle \\ &+ (-p^*qE + qq^*F)|\downarrow\rangle \otimes |-1\rangle - p^*q^*F|\uparrow\rangle \otimes |-1\rangle + p^{*2}F|\downarrow\rangle \otimes |-3\rangle, \end{aligned} \quad (71)$$

and the associated probabilities are

$$\begin{aligned} P_3 &= p^2p^{*2}EE^* = (pp^*)^2EE^* = \frac{1}{2}(pp^*)^2, \quad P_{-3} = (pp^*)^2FF^* = \frac{1}{2}(pp^*)^2, \\ P_1 &= pp^*qq^* + \frac{1}{2}(qq^*)^2, \quad P_{-1} = pp^*qq^* + \frac{1}{2}(qq^*)^2. \end{aligned} \quad (72)$$

Clearly, the above distribution is symmetric. One can continue like this for large times.

Example: Let $p = \frac{\sqrt{3}}{2}$, $q = \frac{1}{2}$, $r = \frac{1}{\sqrt{2}}$ and $s = \frac{i}{\sqrt{2}}$. It can be checked that the constraint (68) holds for this choice. The associated probabilities are

$$\begin{aligned} T = 1 : P_1 &= P_{-1} = 1/2, \\ T = 2 : P_2 &= P_{-2} = 3/8, \quad P_0 = 1/4, \\ T = 3 : P_3 &= P_{-3} = 9/32, \quad P_1 = P_{-1} = 7/32. \end{aligned} \tag{73}$$

Hence we get a new symmetric distribution which is different from the classical one. This example demonstrates new possibilities in the quantum world which have no classical analogues. The distribution depends only on the values of p and q , after the initial constraint (68) is satisfied. For $p = q = \frac{1}{\sqrt{2}}$, we recover the Hadamard walk. So this walk can be thought of as a ‘generalized’ Hadamard walk for unequal amplitudes p and q .

6 Conclusions

In this work, we have established that it is not possible to create a superposition with equal probabilities, of an arbitrary qubit state and its orthogonal. The class of states for which this can be achieved is presented. In addition, by using the principles of no-superluminal signalling and non-increase of entanglement under LOCC we have shown that this is the only set of qubits which would satisfy the equal superposition transformation. In other words, a qubit state and its complement can be equally superposed *if and only if* they belong to the aforementioned ensemble.

The quantum random walk associated with this general unitary equal superposition transformation has been investigated from the point of view of probability distribution of a particle. We have found that the entire family leads to the same asymmetric distribution. This implies that even the symmetric transformation (19) gives an asymmetric walk. It may be mentioned that apart from the CB vectors, any state from our ensemble specified by (10), can be used as a coin state to study the evolution of the walk. The measurement on the coin register would then have to be carried out in the $\{|\psi\rangle, |\bar{\psi}\rangle\}$ basis. We have also obtained conditions under which equal and unequal superpositions would yield unbiased walks. To illustrate this, a few examples have been presented. We have analysed the evolution explicitly only upto four iterations. It would be interesting to simulate the walk associated with the unbalanced coin for large times and study the mixing time and other properties.

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